B. Multiple-Wave Interference

The superposition of M monochromatic waves of the same frequency, with complex amplitudes $U_1, U_2, \ldots, U_M$, gives rise to a wave whose frequency remains the same and whose complex amplitude is given by $U = U_1 + U_2 + \cdots + U_M$. Knowledge of the intensities of the individual waves, $I_1, I_2, \ldots, I_M$, is not sufficient to determine the total intensity $I = |U|^2$ since the relative phases must also be known. The role played by the phase is dramatically illustrated in the following examples.

Interference of M Waves with Equal Amplitudes and Equal Phase Differences

We first examine the interference of M waves with complex amplitudes

$$U_m = \sqrt{I_0}\,\exp[j(m-1)\varphi], \qquad m - 1, 2, \ldots, M\,. \tag{2.5-9}$$

The waves have equal intensities I_0, and phase difference φ between successive waves, as illustrated in Fig. 2.5-7(a). To derive an expression for the intensity of the superposition, it is convenient to introduce the quantity $h = \exp(j\varphi)$ whereupon $U_m = \sqrt{I_0}\,h^{m-1}$. The complex amplitude of the superposed wave is then

$$\begin{aligned} U &= \sqrt{I_0}\,(1 + h + h^2 + \cdots + h^{M-1}) = \sqrt{I_0}\,\frac{1-h^M}{1-h} \\ &= \sqrt{I_0}\,\frac{1-\exp(jM\varphi)}{1-\exp(j\varphi)}\,, \end{aligned} \tag{2.5-10}$$

which has the corresponding intensity

$$I = |U|^2 = I_0 \left| \frac{\exp(-jM\varphi/2) - \exp(jM\varphi/2)}{\exp(-j\varphi/2) - \exp(j\varphi/2)} \right|^2, \tag{2.5-11}$$

whence

$$\boxed{I = I_0 \frac{\sin^2(M\varphi/2)}{\sin^2(\varphi/2)}\,.} \tag{2.5-12}$$

Interference of M Waves

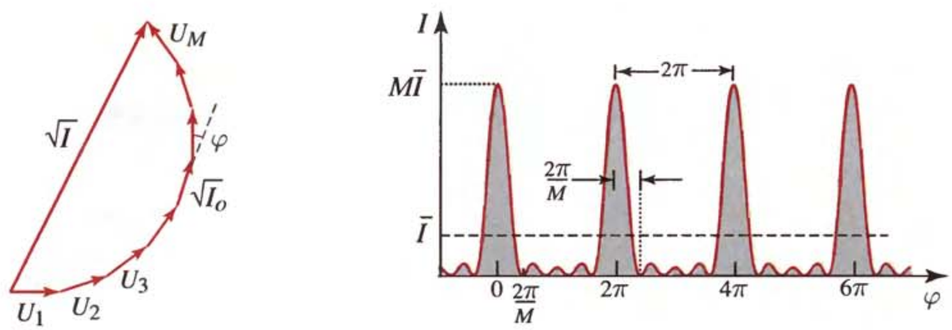

Figure 2.5-7 (a) The sum of M phasors of equal magnitudes and equal phase differences. (b) The intensity I as a function of φ. The peak intensity occurs when all the phasors are aligned; it is then M times greater than the mean intensity $\bar{I} = MI_0$. In this example $M = 5$.

The intensity I is evidently strongly dependent on the phase difference φ, as illustrated in Fig. 2.5-7(b) for $M = 5$. When $\varphi = 2\pi q$, where q is an integer, all the phasors are aligned so that the amplitude of the total wave is M times that of an individual component, and the intensity reaches its peak value of $M^2 I_0$. The mean intensity averaged over a uniform distribution of φ is $\bar{I} = (1/2\pi)\int_0^{2\pi} I\,d\varphi = MI_0$, which is the same as the result obtained in the absence of interference. The peak intensity is therefore M times greater than the mean intensity. The sensitivity of the intensity to the phase is therefore dramatic for large M. At its peak value, the intensity is magnified by a factor M over the mean but it decreases sharply as the phase difference φ deviates slightly from $2\pi q$. In particular, when $\varphi = 2\pi/M$ the intensity becomes zero. It is instructive to compare Fig. 2.5-7(*b*) for $M = 5$ with Fig. 2.5-2 for $M = 2$.

EXERCISE 2.5-3

Bragg Reflection. Consider light reflected at an angle θ from M parallel reflecting planes separated by a distance Λ, as shown in Fig. 2.5-8. Assume that only a small fraction of the light is reflected from each plane, so that the amplitudes of the M reflected waves are approximately equal. Show that the reflected waves have a phase difference $\varphi = k(2\Lambda \sin\theta)$ and that the angle θ at which the intensity of the total reflected light is maximum satisfies

$$\sin\theta = \frac{\lambda}{2\Lambda}. \quad (2.5\text{-}13)$$

Bragg Angle

This equation defines the **Bragg angle** θ. Such reflections are encountered when light is reflected from a multilayer structure (see Sec. 7.1) or when X-ray waves are reflected from atomic planes in crystalline structures. It also occurs when light is reflected from a periodic structure created by an acoustic wave (see Chapter 19). An exact treatment of Bragg reflection is provided in Sec. 7.1C.

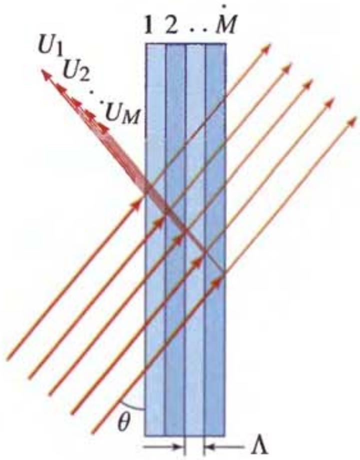

Figure 2.5-8 Reflection of a plane wave from M parallel planes separated from each other by a distance Λ. The reflected waves interfere constructively and yield maximum intensity when the angle θ is the Bragg angle. Note that θ is defined with respect to the parallel planes.