Gaussian beam

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• Very often, we consider scalar wave (obeying Helmholtz equation $\nabla^2 U + k^2 U = 0$) propagating with a small angle along the propagation direction.

• Spherical wave: $U = A_o \exp(-jkr)/r$ is a solution for the Helmholtz equation where A_o is a constant. Good for any distance from the source.

• Paraboloidal wave: When the wave propagates with a small angle the z axis, i.e. r ≈ z, Spherical wave is approximation by a paraboloidal wave

$$U \approx \frac{A_o}{z} \exp(-jkz) \exp(-jk \frac{x^2 + y^2}{2z}).$$

• Characteristics of Gaussian beam (Sect. 3.1):
A generalized paraboloidal wave with nonuniform transverse profile -- in propagation factor $r \approx z(1 + \rho^2/2z^2)$ with z being the propagation distance, $\rho^2 = x^2 + y^2$ and (x,y) on a transverse plane. Gaussian intensity profile is the desirable mode of a laser light.
Important parameters at particular z are spot size (radius) w, radius of curvature R and power P_e.
The above parameters depend on the minimum spot size (at focus, i.e. z=0) w_o, wavelength λ and maximum intensity I_o at z=0 where beam assumes minimum size.

• Paraxial wave equation: slowly varying envelope

approx of wave eqn.
$\nabla_T^2 A - j2k\partial A/\partial z = 0$ where $\nabla_T^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ for rectangular coordinates and A is the complex envelope.

• Gaussian beam solution: It is one of the solution of paraxial wave equation.
$A = A_1 \exp(-jk\rho^2/2q(z))/q(z)$ where
$1/q(z) = 1/(z + jz_o) \equiv 1/R - j\lambda/(\pi w^2)$ is the complex q-parameter and $z_o = \pi w_o^2/\lambda$ is the Rayleigh range measuring diffraction.

• Explicit form of Gaussian beam:
$U = A_o \dfrac{w_o}{w} \exp(-\rho^2/w^2) \exp(-jkz - jk\rho^2/(2R) + \tan^{-1}(z/z_o))$
where $w = w_o\sqrt{1 + (z/z_o)^2}$ and $R = z(1 + (z_o/z)^2)$.
Notes -- $|U| = |E|/\sqrt{2\eta}$ and
At z = 0, we have min. spot size w_o, $R \to \infty$,
$I_o = |A_o|^2$;
At $z = z_o$, $w = \sqrt{2}w_o$ and $R = 2z$; at $z \to \infty$, $w \to \infty$,
$R \to \infty$.

• Important parameters:
w is defined at the $I = I_{max}e^{-2}$,
$P_e = I_o\pi w_o^2/2$,
divergence angle $2\theta_o = 2\lambda/(\pi w_o)$.
depth of focus $2z_o$.

• Characterize a Gaussian beam:
$q = z + jz_o$

Observations: $z_2 = z_1 + d \rightarrow q(z_2) = q(z_1) + d$, i.e. we can easily find R_2 & w_2 from R_1 & w_1 without knowing w_o and z_1.
Notice that our reference is the focus which the origin $\rightarrow$ At focus, $q = jz_o$ purely imaginary.

• Other beam solutions for paraxial wave equation: Hermite-Gaussian beams (HG) (sec 3.3), and Lagurrre-Gaussian (LG) and Bessel beams (sec 3.4).
Notes -- High order Lagurrre-Gaussian and Bessel beams carries angular momentum

• Scalar wave is good for TEM EM wave (e.g. plane wave) that has only transverse field component. As result, we can neglect the vectorial nature or polarization effect.

• Spherical EM wave: Consider a short dipole antenna with vector potential $\vec{A} = \hat{x}U(\vec{r})A_o$ where $U(\vec{r})$ is the scalar spherical wave solution.
$\vec{H} = \nabla \times \vec{A}/\mu_o$, $\vec{E} = \nabla \times \vec{H}/(j\omega\varepsilon)$
$\vec{H} = H_o \sin\theta U(\vec{r})(1 + (jkr)^{-1})\hat{\phi}$
For $r \gg \lambda$, $\vec{H} \approx H_o \sin\theta U(\vec{r})\hat{\phi}$ and $\vec{E} \approx E_o \sin\theta U(\vec{r})\hat{\theta}$
For paraxial wave, $\vec{E} = E_o(-\hat{x} + \hat{z}x/z)U(\vec{r})$
Note: There is longitudinal field component (along z).

HW #2.1 due 2/11/14

1. Problem 5.2-1 (page 191 1st Ed, p. 195 2nd Ed)
2. Problem 5.3-1 (page 191 1st Ed, p. 195 2nd Ed)