HW 2

1) $\vec{E} = \hat{y}\, 5\cos(2\pi\times10^{10} t + 200\pi x)$, $\mu = \mu_0$

a) $\omega = 2\pi\times10^{10}$ (rad/s) $\Rightarrow f = 10^{10}$ (Hz)

Polarization $-\ y$

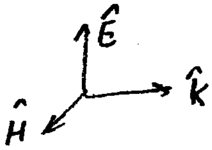
$\vec{K}$ = wave vector $= -\hat{x}\, 200\pi$

b) $u_p = \dfrac{\omega}{K} = \dfrac{2\pi\times10^{10}}{200\pi} = 10^{8}$ (m/s)

$u_p = \dfrac{1}{\sqrt{\mu\epsilon}} \Rightarrow \epsilon = \dfrac{1}{\mu u_p^2} = \dfrac{1}{4\pi\times10^{-7}\ 10^{16}} = 7.958\times10^{-11}$ (F/m)

$= 3^2\epsilon_0$ (i.e. $n = 3$)

c) $\vec{H} = \dfrac{|\vec{E}|}{\eta}\, \hat{k}\times\hat{E}\ e^{j200\pi x}$



$= \dfrac{5}{377/3}\ -\hat{x}\times\hat{y}\ e^{j200\pi x} = \hat{z}\,0.03979\, e^{j200\pi x}$ (A/m)

d) $\langle \vec{S} \rangle = \hat{k}\,\dfrac{1}{2}\,\dfrac{|E|^2}{\eta} = -\hat{x}\,\dfrac{1}{2}\,\dfrac{|5|^2}{377/3} = -\hat{x}\,0.09947$ (W/m^2)

e) Power $= |\vec{S}_{ave}| \times$ area $= \pi\, 2^2 \times 0.09947$

$= 1.25$ (W)

2) a) (5.5-5) $\quad n - j\dfrac{1}{2}\dfrac{\alpha}{k_0} = \sqrt{1+\chi' + j\chi''}$

$\chi'' \ll 1+\chi' \qquad \sqrt{1+\chi'+j\chi''} = \sqrt{1+\chi'}\,\sqrt{1+\dfrac{j\chi''}{1+\chi'}}$

$\simeq \sqrt{1+\chi'}\left(1+\dfrac{1}{2}\dfrac{j\chi''}{1+\chi'}\right)$

$= \sqrt{1+\chi'} + \dfrac{1}{2}\dfrac{j\chi''}{\sqrt{1+\chi'}}$

Matching real part: $n = \sqrt{1+\chi'}$ (5.5-7)

matching imaginary part: $-\dfrac{1}{2}\dfrac{\alpha}{k_0} = \dfrac{1}{2}\dfrac{\chi''}{\sqrt{1+\chi'}}$

$\Rightarrow \alpha = -k_0\chi''/n$ (5.5-8)

$$\eta = \frac{\eta_0}{n - j\frac{1}{2}\frac{\alpha}{k_0}} = \frac{\eta_0}{n + \frac{1}{2}\frac{\chi''}{n}j} \times \frac{n - \frac{1}{2}\chi''/n\,j}{n - \frac{1}{2}\frac{\chi''}{n}j} = \frac{n\eta_0 - \frac{\eta_0}{2}\frac{\chi''}{n}j}{n^2 + \frac{1}{4}\left(\frac{\chi''}{n}\right)^2}$$

$$Re(\eta) = \frac{n\eta_0}{n^2 + \frac{1}{4}\left(\frac{\chi''}{n}\right)^2} \qquad Im(\eta) = \frac{-\chi''}{2n\left(n^2 + \frac{1}{4}\left(\frac{\chi''}{n}\right)^2\right)}\eta_0$$

b) $\chi'' >> |1+\chi'|$ $\sqrt{1+\chi'+j\chi''}$

$$\simeq \sqrt{j\chi''} = \sqrt{-j(-\chi'')} \quad \text{(since we know } \chi''<0)$$

$$= \pm\left[e^{-j\frac{\pi}{2}}\right]^{\frac{1}{2}}\sqrt{-\chi''} = \pm\frac{1}{\sqrt{2}}(1-j)\sqrt{-\chi''}$$

We choose + to keep imaginary part negative.

Real part $n = \sqrt{\frac{-\chi''}{2}}$

Imaginary part $-j\frac{1}{2}\frac{\alpha}{k_0} = -\frac{j}{\sqrt{2}}\sqrt{-\chi''}$

$$\alpha = 2k_0\sqrt{-\chi''/2}$$

$$\eta = \frac{\eta_0}{\frac{1}{\sqrt{2}}\sqrt{-\chi''}(1-j)}\frac{1+j}{1+j} = \frac{\eta_0(1+j)}{\sqrt{-2\chi''}}$$

$$Re(\eta) = \frac{\eta_0}{\sqrt{-2\chi''}} \qquad Im(\eta) = \frac{\eta_0}{\sqrt{-2\chi''}}$$

3) $1+\chi_h = n_0^2$

$$n - j\frac{1}{2}\frac{\alpha}{k_0} = \sqrt{1+\chi_h+\chi'+j\chi''} = \sqrt{n_0^2+\chi'+j\chi''} = n_0\sqrt{1+\frac{\chi'+j\chi''}{n_0^2}}$$

$$\simeq n_0\left(1+\frac{1}{2}\frac{\chi'+j\chi''}{n_0^2}\right) = n_0 + \frac{1}{2}\frac{\chi'}{n_0} + j\frac{\chi''}{2n_0}$$

Real part: $n = n_0 + \frac{1}{2}\frac{\chi'}{n_0}$ Imaginary part: $\alpha = -k_0\chi''/n_0$

4) Maxwell's equations for free space:

$$\nabla\times\vec{H} = j\omega\epsilon_0\vec{E} \quad — ①$$

$$\nabla\times\vec{E} = -j\omega\mu_0\vec{H} \quad — ②$$

assume $\vec{H} = \vec{\mathcal{H}}\exp(-j\vec{k}\cdot\vec{r})$ — ③

$\vec{E} = \vec{\mathcal{E}}\exp(-j\vec{k}\cdot\vec{r})$ — ④

And $\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$

$\vec{\mathcal{H}} = \hat{x}H_x + \hat{y}H_y + \hat{z}H_z$

$$\vec{\mathcal{E}} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$$

You can find the algebraic from of Maxwell's equations by substituting (3),(4) into (1):

$$\nabla \times \left(\vec{H} \exp(-j\vec{k}\cdot\vec{r})\right) = j\omega\epsilon_0 \vec{E}$$

$$\nabla \exp(-j\vec{k}\cdot\vec{r}) \times \vec{H} + \exp(-j\vec{k}\cdot\vec{r}) \underbrace{\cancel{\nabla\times\vec{H}}}_{0} = j\omega\epsilon_0 \vec{E}$$

$$-j\vec{k}\times\vec{H}\exp(-j\vec{k}\cdot\vec{r}) = j\omega\epsilon_0 \vec{\mathcal{E}}\exp(-j\vec{k}\cdot\vec{r})$$

$$\vec{k}\times\vec{H} = -\omega\epsilon_0\vec{\mathcal{E}} \,//$$

Substituting (3),(4) into (2):

$$\nabla \times \left(\vec{\mathcal{E}}\exp(-j\vec{k}\cdot\vec{r})\right) = -j\omega\mu_0 \vec{H}\exp(-j\vec{k}\cdot\vec{r})$$

$$\nabla \exp(-j\vec{k}\cdot\vec{r}) \times \vec{\mathcal{E}} + \exp(-j\vec{k}\cdot\vec{r}) \underbrace{\cancel{\nabla\times\vec{E}}}_{0} = -j\omega\mu_0\vec{H}\exp(-j\vec{k}\cdot\vec{r})$$

$$\vec{k}\times\vec{\mathcal{E}} = \omega\mu_0\vec{H} \,//$$

Extra-credit problem

a)

$$\vec{k}\times\vec{H} = -\omega\vec{D}$$

$$\vec{k}\cdot(\vec{k}\times\vec{H}) = -\omega\,\vec{k}\cdot\vec{D}$$

$$\vec{H}\cdot\underbrace{(\cancel{\vec{k}\times\vec{k}})}_{0} = -\omega\,\vec{k}\times\vec{D}$$

$$\vec{k}\cdot\vec{D} = 0 \,//$$

b)

$$\vec{k}\times\vec{H} = -\omega\vec{D}$$

$$(\vec{k}\times\vec{H})\times\vec{B} = -\omega\vec{D}\times\vec{B}$$

$$\vec{B}\times(\vec{k}\times\vec{H}) = \omega\vec{D}\times\vec{B}$$

$$\vec{k}(\vec{B}\cdot\vec{H}) - \vec{H}(\vec{B}\cdot\vec{k}) = \omega\vec{D}\times\vec{B}$$

Note: $\vec{k}\cdot(\vec{k}\times\vec{E}) = \omega\vec{k}\cdot\vec{B} \Rightarrow \vec{k}\cdot\vec{B} = 0$ (like part a)

$$\vec{D}\times\vec{B} = \frac{\vec{k}}{\omega}(\vec{B}\cdot\vec{H}) \,//$$

$\therefore$ $\vec{D}\times\vec{B}$ is in $\vec{k}$ direction

c)

$$\vec{k}\times\vec{H} = -\omega\vec{D}$$

$$\frac{1}{\mu_0}\vec{k}\times\vec{B} = -\omega\vec{D}$$

$$\frac{1}{\mu_0}\vec{k}\times(\vec{k}\times\vec{E}) = -\omega^2\vec{P}$$

$$\vec{k}(\vec{k}\cdot\vec{E}) - \vec{E}\,|\vec{k}|^2 = -\omega^2\mu_0\vec{D}$$

$$\underbrace{\cancel{\vec{D}\cdot\vec{k}}}_{0}(\vec{k}\cdot\vec{E}) - \vec{D}\cdot\vec{E}\,|\vec{k}|^2 = -\omega^2\mu_0\vec{D}\cdot\vec{D}$$

$$|\vec{k}|^2 = \omega^2\mu_0\frac{\vec{D}\cdot\vec{D}}{\vec{E}\cdot\vec{D}} \quad //$$

d) $\vec{H} = \frac{1}{\mu_0}\vec{B}$ $\quad \vec{D} = \epsilon_0\vec{E} + \vec{P} \Rightarrow \vec{E} = \frac{1}{\epsilon_0}(\vec{D} - \vec{P})$

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2\mu_0\epsilon_0}(\vec{D}\times\vec{B}^* - \vec{P}\times\vec{B}^*)$$

From b) we know $\vec{D}\times\vec{B}^* = \frac{\vec{k}}{\omega}(\vec{B}^*\cdot\vec{H})$. However, $\vec{P}\times\vec{B}^*$ may not be in the direction of $\vec{k}$.