HW4 Solution

1) Given path length $L = n_1 d_1 \sec\theta_1 + n_2 d_2 \sec\theta_2$ — ①

and $d_1 \tan\theta_1 + d_2 \tan\theta_2 = d$ — ②

Differentiate ②: $d_1 \sec^2\theta_1 \, d\theta_1 + d_2 \sec^2\theta_2 \, d\theta_2 = 0$

$$\Rightarrow \frac{d\theta_2}{d\theta_1} = -\frac{d_1 \sec^2\theta_1}{d_2 \sec^2\theta_2} \quad — ③$$

Differentiate ① with respect to θ_1: $n_1 d_1 \sec\theta_1 \tan\theta_1 + n_2 d_2 \sec\theta_2 \tan\theta_2 \dfrac{d\theta_2}{d\theta_1} = \dfrac{dL}{d\theta_1}$

For minimization: $\dfrac{dL}{d\theta_1} = 0$ and substitute ③ into equation

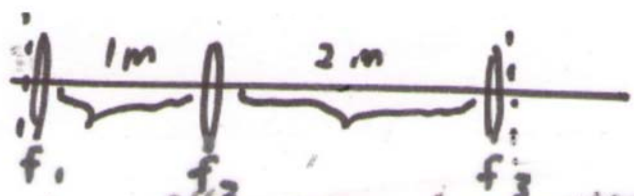
$$n_1 d_1 \sec\theta_1 \tan\theta_1 - n_2 \cancel{d_2} \sec\theta_2 \tan\theta_2 \frac{d_1 \sec^2\theta_1}{\cancel{d_2} \sec^2\theta_2} = 0$$

$$\Rightarrow d_1 \sec\theta_1 \left(n_1 \tan\theta_1 - n_2 \frac{\tan\theta_2}{\sec\theta_2} \sec\theta_1\right) = 0$$

$$\Rightarrow d_1 \frac{\sec\theta_1}{\cos\theta_1} (n_1 \sin\theta_1 - n_2 \sin\theta_2) = 0 \Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Snell's law

2) a)



len_3 w/ f_3 2m len_2 w/ f_2 1m len_1 w/ f_1

$$\text{ray matrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.6} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.25} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.5} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.6} & 1 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.6} & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -3 & 5\frac{1}{3} \end{bmatrix}$$

Note: Diagram has elements going from left to right.
Matrix is written from right to left.

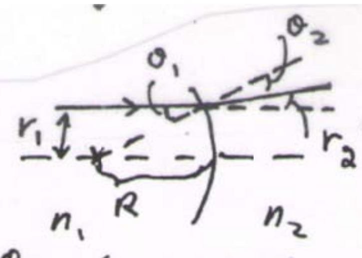
b) $r = r_0 \cos\alpha z + \frac{r_0'}{\alpha} \sin\alpha z$

$r' = -r_0\alpha \sin\alpha z + r_0' \cos\alpha z$

$\underline{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\alpha z & \frac{1}{\alpha}\sin\alpha z \\ -\alpha\sin\alpha z & \cos\alpha z \end{bmatrix}$ where $z = d$ length of fiber.

Check $Det(\underline{M}) = 1$. Notice that this matrix for inside the fiber.

3) Consider $r_1 \neq 0$, $r_1' = 0$



$r_2 = r_1 \quad -①$

$r_2' = \theta_1 - \theta_2$ with $n_1 \sin\theta_1 = n_2 \sin\theta_2$ (snell's law)

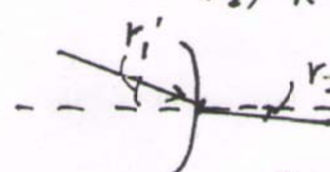
$\Rightarrow \theta_2 \simeq \frac{n_1}{n_2}\theta_1$

$r_2' = (1 - n_1/n_2)\theta_1 = (1 - \frac{n_1}{n_2})\frac{r_1}{R} \quad -②$

Compare ① to $r_2 = Ar_1 + B\cancel{r_1'}^{\,0} \Rightarrow A = 1$ //

compare ② to $r_2' = Cr_1 + D\cancel{r_1'}^{\,0} \Rightarrow C = (1 - \frac{n_1}{n_2})\frac{1}{R}$ //

Consider $r_1' \neq 0$, $r_1 = 0$



$r_2 = 0 \quad -③$

$n_1 r_1' = n_2 r_2'$ (Snell's law)

$\Rightarrow r_2' = \frac{n_1}{n_2} r_1' \quad -④$

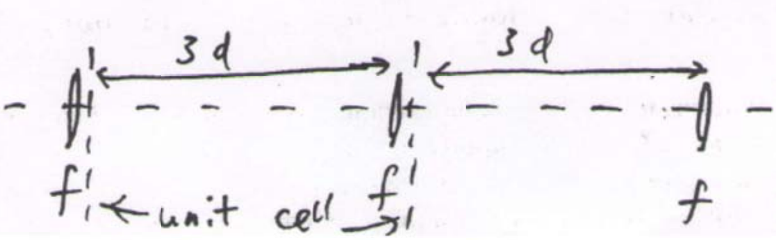
$\underline{M} = \begin{bmatrix} 1 & 0 \\ (1 - \frac{n_1}{n_2})\frac{1}{R} & n_1/n_2 \end{bmatrix}$

Compare ③ to $r_2 = A\cancel{r_1}^{\,0} + Br_1' \Rightarrow B = 0$ //

Compare ④ to $r_2' = C\cancel{r_1}^{\,0} + Dr_1' \Rightarrow D = n_1/n_2$ //

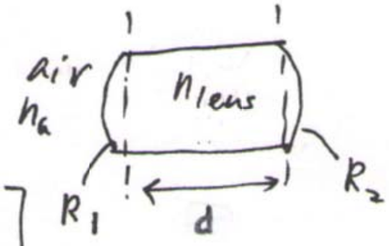
Note: C has opposite of the convex boundary derived in class.

4) a)



b) $\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3d \\ -\frac{1}{f} & 1 - \frac{3d}{f} \end{bmatrix}$

5) Extra-Credit for undergradutte



a) Ray matrix $= \begin{bmatrix} 1 & 0 \\ \left(1-\frac{n_{lens}}{n_a}\right)\frac{1}{R_2} & \frac{n_{lens}}{n_a} \end{bmatrix}$

$$\times \underbrace{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\left(1-\frac{n_a}{n_{lens}}\right)\frac{1}{R_1} & \frac{n_a}{n_{lens}} \end{bmatrix}}_{\begin{bmatrix} 1-\left(1-\frac{n_a}{n_{lens}}\right)\frac{d}{R_1} & \frac{n_a}{n_{lens}} d \\ -\left(1-\frac{n_a}{n_{lens}}\right)\frac{1}{R_1} & n_a/n_{lens} \end{bmatrix}}$$

$$= \begin{bmatrix} 1-\left(1-\frac{n_a}{n_{lens}}\right)\frac{d}{R_1} & \frac{n_a}{n_{lens}} d \\ \left(1-\frac{n_{lens}}{n_a}\right)\left[\frac{1}{R_1}+\frac{1}{R_2}-\frac{d}{R_1R_2}\left(1-\frac{n_a}{n_{lens}}\right)\right] & \left(\frac{n_a}{n_{lens}}-1\right)\frac{d}{R_2}+1 \end{bmatrix}$$

c) Compare $C=-1/f$ to lensmaker formula : $\frac{n_a}{f} = (n_{lens} - n_a)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$

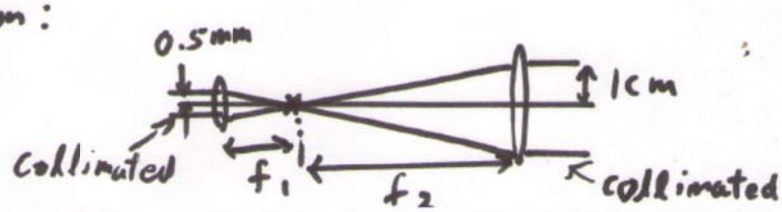
The C in ray matrix is a modified version of lensmaker formula. They are the same

b) if $d=0$! More precisely, $\frac{n_a}{f} = (n_{lens}-n_a)\left[\frac{1}{R_1}+\frac{1}{R_2}-\frac{d}{R_1R_2}\left(1-\frac{n_a}{n_{lens}}\right)\right]$

6) Extra Credit

This is a system with magnification of 20. You can draw the ray diagram:

$$\frac{f_2}{f_1} = 20$$



Argon Ion is a visible laser (HW 1). There is no requirement on loss. We can use any ordinary lens without antireflective coating. Look up Edmund Optics catalog

For lens 1, we use 532-023 (\$35) 3mm diameter & 3mm focal length.

For lens 2, we use B 45-167 (\$38.5) 4cm diameter & 60mm focal length.

Note: We use lens size at least 4 times of beam size.

Total cost: \$73.5 for lens

You may need other parts e.g. lens mounts & casing.

$$\underline{M} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.06} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.063 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.003} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.063 \\ -\frac{1}{0.06} & 1-\frac{0.063}{0.06} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{0.003} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & 0.063 \\ 0 & -0.05 \end{bmatrix}$$

input ray $\begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \begin{bmatrix} 0.0005 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \underline{M} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix} = \begin{bmatrix} -0.01 \\ 0 \end{bmatrix}$$

This matches with the ray diagram.