HW 7 Solution

1) $\lambda_0 = 514.5\,nm$, $W_0 = 0.6\,mm.$, air $\Rightarrow n = 1$

a) $W(z) = 0.8\,mm$, $z = ?$, $R(z) = ?$

$$W(z) = W_0\sqrt{1+\left(\frac{z}{z_0}\right)^2} \quad , \quad z_0 = \frac{n\pi W_0^2}{\lambda_0} = 2.198\,(m)$$

$$\Rightarrow \frac{z}{z_0} = \sqrt{\left(\frac{W(z)}{W_0}\right)^2 - 1} = \sqrt{\left(\frac{0.8}{0.6}\right)^2 - 1} \Rightarrow z = 1.938\,(m)$$

$$R = z\left(1+\left(\frac{z_0}{z}\right)^2\right) = 4.431\,(m)$$

q=1.938+j2.198

b) Exercise 3.1-4

$\lambda = 1\,\mu m$, $W_1 = 1\,mm$, $R_1 = 1\,m$. $d = 0.1\,m$.

$$\frac{1}{q_1} = \frac{1}{R_1} - j\frac{\lambda}{\pi W_1^2} = 1 - j\frac{10^{-6}}{\pi 10^{-6}} = 1 - j\frac{1}{\pi}$$

$$q_1 = \frac{1+j\frac{1}{\pi}}{1+\left(\frac{1}{\pi}\right)^2} = \frac{1}{1.101} + j0.289 \quad , \quad q_2 = q_1 + 0.1$$

$$\frac{1}{q_2} = \frac{1.008 - j0.289}{(1.008)^2 + 0.289^2} = \frac{1.008}{1.1} - j0.2628$$

$$\frac{1}{R_2} = \frac{1.008}{1.1} \Rightarrow R_2 = 1.091\,(m)$$

$$\frac{\lambda}{\pi W_2^2} = 0.2628 \Rightarrow W_2 = \sqrt{\frac{\lambda}{0.2628\,\pi}} = 1.1\times10^{-3}\,(m) = 1.1\,(mm)$$

2a) $\lambda_0 = 1.06\,\mu m$, $P = 1\,W$, $2\theta_0 = 10^{-3}\,rad$ $n=1$

$$2\theta_0 = \frac{2\lambda_0}{\pi W_0} \Rightarrow W_0 = \frac{2\lambda_0}{\pi(2\theta_0)} = \frac{2\times1.06\times10^{-6}}{\pi\,10^{-3}} = 6.748\times10^{-4}\,(m)$$

$$P = I_0\pi W_0^2/2 \Rightarrow I_0 = \left[\frac{\pi W_0^2}{2P}\right]^{-1} = \left[\frac{\pi\times(6.748\times10^{-4})^2}{2}\right]^{-1} = 1.398\,MW/m$$

$$I(z=100\,cm) = I_0\left(\frac{W_0}{W}\right)^2 = I_0 \Big/ \left[1+\left(\frac{1}{1.35}\right)^2\right] = 0.9027\,MW/m^2)$$

$$z_0 = \frac{\pi W_0^2}{\lambda_0} = \frac{\pi(6.748\times10^{-4})^2}{1.06\times10^{-6}} = 1.35\,(m)$$

depth of focus $= 2z_0 = 2.7\,(m)$

2b) Exercise 3.2-6

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & d/n \\ 0 & 1/n \end{bmatrix} = \begin{bmatrix} 1 & d/n \\ 0 & 1 \end{bmatrix}$$

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} = q_1 + d/n \Rightarrow \frac{1}{q_2} = \frac{1}{q_1 + d/n}$$

$$\Rightarrow \frac{1}{R_2} - j\frac{\lambda}{\pi W_2^2} = \frac{1}{z + jz_0 + d/n} = \frac{(z + d/n) - jz_0}{(z + \frac{d}{n})^2 + z_0^2}$$

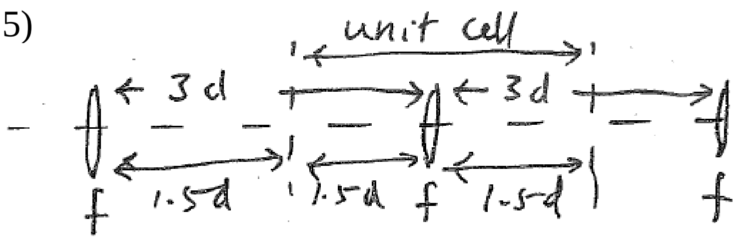
$$\frac{1}{R_2} = \frac{z + d/n}{(z + \frac{d}{n})^2 + z_0^2} \Rightarrow R_2 = \left(z + \frac{d}{n}\right)\left(1 + \left(\frac{z_0}{z + d/n}\right)^2\right)$$

$$\frac{\lambda}{\pi W_2^2} = \frac{z_0}{(z + d/n)^2 + z_0^2} \Rightarrow W_2^2 = \frac{\lambda}{\pi} z_0^2 \left(1 + \frac{(z + d/n)^2}{z_0^2}\right) = W_0^2\left(1 + \left(\frac{z + d/n}{z_0}\right)^2\right)$$

With the layer with different refractive index from air, the parameters of the beam behave like traveling a short distance of $z + d/n$ instead of the actual distance of $z + d$. This is owing to the reduced speed in the layer.

Extra-credit 5)

a)



b) Since lenses are identical, the z_{min} must be in the middle of the space, i.e. 1.5d.

We can verify this by the cell unit shown in a)

$$\begin{bmatrix} 1 & 1.5d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1.5d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1.5d}{f} & 1.5d\left(1 - \frac{1.5d}{f}\right) \\ -\frac{1}{f} & 1 - \frac{1.5d}{f} \end{bmatrix}$$

Use the formula $\frac{1}{q} = \frac{D-A}{2B} - j\frac{1}{B}\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}$

At z_{min}, $D=A$ $(R \to \infty)$. The ray matrix of the cell has this property. So our expectation is correct.

c) At the z_{min} $-Im\left(\frac{1}{q}\right) = \frac{\lambda}{\pi W_0^2}$

$$\Rightarrow \frac{1}{B}\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}} = \frac{\lambda}{\pi W_0^2}$$

$$\Rightarrow W_0^2 = \frac{\lambda B}{\pi\left[1-\left(\frac{A+D}{2}\right)^2\right]^{\frac{1}{2}}} = \frac{\lambda B}{\pi\left[1-A^2\right]^{\frac{1}{2}}} \quad (\text{since } A=D)$$

$$W_0 = \sqrt{-\frac{\lambda}{\pi}\frac{(1-A^2)^{\frac{1}{2}}}{C}} = \left[\frac{\lambda}{\pi}\sqrt{3df\left(1-\frac{3d}{4f}\right)}\right]^{\frac{1}{2}}$$

$(AD-BC=1 \Rightarrow A^2-BC=1)$

d) $\left|\frac{A+D}{2}\right| < 1$ in order for $\sqrt{1-\left(\frac{A+D}{2}\right)^2}$ to be real.
(or $\sqrt{1-A^2}$)

Otherwise, W_0 is an imaginary number.

Therefore all formulas in this problem is for stable cavity.