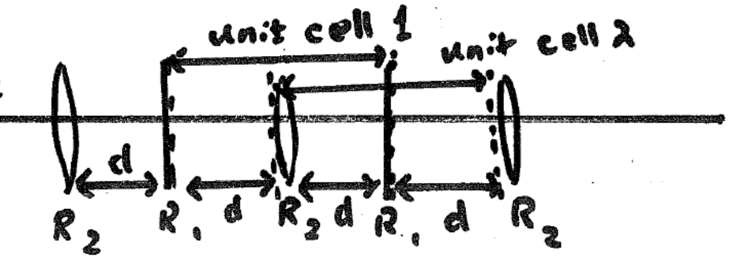
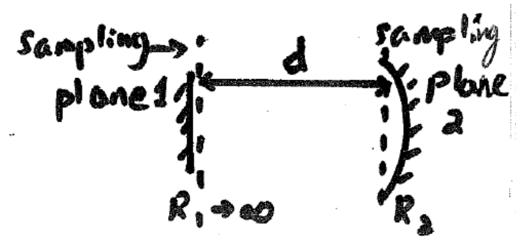
HW 8 Solution

1)
Exercise 10.2-2





$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ -\frac{1}{f_2} & 1-\frac{d}{f_2} \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f_2} & d\left(2-\frac{d}{f}\right) \\ -\frac{1}{f_2} & 1-\frac{d}{f_2} \end{bmatrix}$$

stable if $\left|\frac{A+D}{2}\right| \leq 1 \Rightarrow \left|1-\frac{d}{f_2}\right| \leq 1 \Rightarrow \left|1-\frac{2d}{|R_2|}\right| \leq 1$ mirror 2 must be converging.

A better way is to find the unit cell 2.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 2d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} = \begin{bmatrix} 1-2d/f_2 & 2d \\ -1/f_2 & 1 \end{bmatrix}$$

$\left|\frac{A+D}{2}\right| \leq 1 \Rightarrow \left|1-d/f_2\right| \leq 1 \Rightarrow \left|1-\frac{2d}{|R_2|}\right| \leq 1$ stable.

From Ch. 10 & lecture. We know R from Gaussian beam must fit radius of curvature of the mirror, i.e.

$$R_2 = d\left(1+(z_0/d)^2\right) \Rightarrow (z_0/d)^2 = \frac{R_2}{d}-1 \Rightarrow z_0 = d\sqrt{\frac{R_2}{d}-1}$$

$$\text{Depth of focus} = 2z_0 = 2d\sqrt{\frac{R_2}{d}-1} = 2d\sqrt{\frac{1}{(d/|R_2|)}-1}$$

$$W_2^2 = W_0^2\left(1+\left(\frac{d}{z_0}\right)^2\right) \Rightarrow W_2^2 = \frac{\lambda}{\pi}z_0\left(1+\frac{1}{\frac{R_2}{d}-1}\right) = \frac{\lambda}{\pi}d\sqrt{\frac{R_2}{d}-1}\left[1+\frac{1}{\frac{R_2}{d}-1}\right]$$

let $x = d/|R_2|$, $W_2 = \left(\frac{1}{x}-1\right)^{\frac{1}{4}}\left[\frac{\lambda}{\pi}d\left(1+\frac{1}{\frac{1}{x}-1}\right)\right]^{\frac{1}{2}}$

$$\Rightarrow W_2 = \left(\frac{1-x}{x}\right)^{\frac{1}{4}}\left[\frac{\lambda}{\pi}d\right]^{\frac{1}{2}}\left(\frac{1}{1-x}\right)^{\frac{1}{2}} = \left[\frac{\lambda}{\pi}d\right]^{\frac{1}{2}}\left(\frac{1}{x(1-x)}\right)^{\frac{1}{4}}$$

or from lecture, we know

$$\frac{\pi W_2^2}{\lambda} = \frac{B}{\sqrt{1-\left(\frac{A+D}{2}\right)^2}} \text{ for cell 2}, \quad \frac{\pi W_0^2}{\lambda} = \frac{B}{\sqrt{1+\left(\frac{A+D}{2}\right)^2}} \text{ for cell 1}$$

(note: A, B, D from 2nd $\underline{M}$) (note: A, B, D from 1st $\underline{M}$)

we obtain the same equations for W_2 and z_0.

2)
Exercise 10.2-3 $d = 30\text{cm}$, confocal, $n=1$

confocal $\Rightarrow$ $d=R$, focus at the center

z_2 = distance between M_2 & center $= \frac{d}{2} = \frac{R}{2}$

$R_2 = R \Rightarrow z_2\left(1+\left(\frac{z_0}{z_2}\right)^2\right) = R \Rightarrow \frac{1}{2}\left(1+\left(\frac{z_0}{z_2}\right)^2\right) = 1 \Rightarrow z_0 = z_2$

similarly, $z_1 = -z_0$

Longitinal mode spacing $= \Delta\nu_{FSR} = \frac{c}{2nd}$

transverse mode spacing $= \Delta\nu_{FSR}\frac{\Delta(m+n)}{\pi}\left[\tan^{-1}\left(\frac{z_2}{z_0}\right) - \tan^{-1}\left(\frac{z_1}{z_0}\right)\right]$

$\Delta\nu_{FSR} = \frac{3\times10^8}{2\times30\times10^{-2}} = 5\times10^8\,(Hz)$

Now $\nu = 5\times10^{14}\,(Hz) \Rightarrow p = 10^6$

$\nu \pm 2\times10^9\,(Hz) \Rightarrow p = 10^6 \pm 4$

If $\Delta(m+n) = 1$, transverse mode spacing

$= \frac{\Delta\nu_{FSR}}{\pi}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$ (recall $z_2 = z_0$, $z_1 = -z_0$)

$= \Delta\nu_{FSR}/2$. Hence, we consider $p = 10^6 \pm 4$ every 0.5 increment, i.e. 5×10^{14}, $5\times10^{14}\pm2.5\times10^8$, $5\times10^{14}\pm5\times10^8$, $5\times10^{14}\pm7.5\times10^8$, $5\times10^{14}\pm10^9$, $5\times10^{14}\pm12.5\times10^8$, $5\times10^{14}\pm15\times10^8$, $5\times10^{14}\pm17.5\times10^8$, $5\times10^{14}\pm2\times10^9$.

3) Confocal, $d = 16cm$, $\mathcal{R}_1 = \mathcal{R}_2 = 0.995$, $n = 1$, $\lambda_0 = 1\mu m$.

a) confocal, we show in 2) $z_1 = -z_0$, $z_2 = z_0$. Problem 10.2-11

$R = d = 16cm$.

b) $z_2 = z_0 \Rightarrow z_0 = d/2 \Rightarrow \frac{\pi W_0^2}{\lambda} = \frac{d}{2} \Rightarrow W_0 = \sqrt{\frac{\lambda}{\pi}\frac{d}{2}} = \sqrt{\frac{10^{-6}\,16\times10^{-2}}{2\pi}} = 0.1596$ (mm)

($z_2 = d/2$)

d) Transverse spacing $\Delta\nu = \frac{\Delta\nu_{FSR}}{\pi}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = \Delta\nu_{FSR}/2 = 4.688\times10^8\,(Hz)$

0,0 mode $\nu = p\,\Delta\nu_{FSR} + (0+0+1)\Delta\nu = \Delta\nu_{FSR}(p+\frac{1}{2}) = 3\times10^{14} + 4.688\times10^8\,(Hz)$

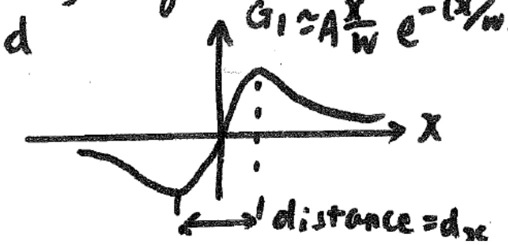
($\frac{\lambda_0}{2}p = d \Rightarrow p = 320000$)

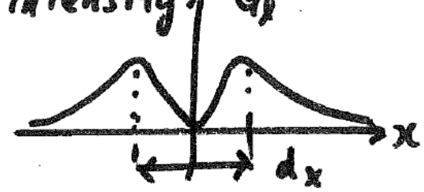
1,0 mode $\nu = p\,\Delta\nu_{FSR} + (1+0+1)\Delta\nu = 3\times10^{14} + 9.376\times10^8\,(Hz)$

e) $\alpha_r = \frac{1}{2d}\ln\left(\frac{1}{\mathcal{R}_1\mathcal{R}_2}\right) = \frac{1}{2(0.16)}\ln\left(\frac{1}{0.995^2}\right) = 0.03133\,(m^{-1})$

c) TEM_{10} has a dip in x while the Gaussian distribution remains in y direction. From Fig. 3.3-1 on p.96.

Field $G_1 \approx A\frac{x}{W}e^{-(x/W)^2}$ intensity G_1^2





To find d_x, we need to locations (in x) of G_1's extreme values. Let $X = \frac{x}{w}$, $G_1 = A X e^{-X^2}$

$$\frac{dG_1}{dX} = 0 \Rightarrow e^{-X^2} + X e^{-X^2}(-2X) = 0 \Rightarrow 1 - 2X^2 = 0 \quad X = \pm\frac{1}{\sqrt{2}}$$

Therefore, $x = \pm\frac{1}{\sqrt{2}} W \Rightarrow d_x = \frac{2}{\sqrt{2}} W = \sqrt{2} W$

At the mirror, $W = \sqrt{2} W_0$ (from $z_2 = z_0$) $\Rightarrow d_x = 2W_0 = 0.3192$ (mm)

4)

a) $d = 0.75\,m$, $R_1 = \infty$, $R_2 = 1\,m$, $\lambda_0 = 632.8\,nm$, $D = 4\,mm$

This is a standing wave cavity. We can use g_1 & g_2

$g_1 = (1 - \frac{d}{R_1}) = 1$ $\quad g_2 = (1 - \frac{d}{R_2}) = 1 - 0.75 = 0.25$

$g_1 g_2 = 0.25$ which is $0 \le g_1 g_2 \le 1 \Rightarrow$ stable.

b) Since the cavity is stable, we can assume the radius of curvature of the Gaussian beam fits the mirrors. At the flat mirror, $W = W_0$ and $R_1 = \infty$.

At the curve mirror, $R_2 = 1$, $W_2 = W_0 \sqrt{1 + (z/z_0)^2}\Big|_{z=d}$.

$$R_2 = z\left(1 + \left(\frac{z_0}{z}\right)^2\right)\Big|_{z=d} \Rightarrow z_0 = z\sqrt{\frac{R_2}{z} - 1}\Big|_{z=d} = 0.75\sqrt{\frac{1}{0.75} - 1} = 0.433$$

$$z_0 = \frac{\pi W_0^2}{\lambda} = \frac{\pi W_0^2}{\lambda_0} \text{ (Assume } n=1 \text{) in He and Ne gases} \Rightarrow W_0 = \sqrt{\frac{z_0 \lambda_0}{\pi}} = 2.953 \times 10^{-4} \text{ (m)}$$

c) $$W_2 = W_0 \sqrt{1 + (z/z_0)^2}\Big|_{z=d} = 2.953\times10^{-4}\sqrt{1 + \left(\frac{0.75}{0.433}\right)^2} = 5.906\times10^{-4} \text{ (m)}$$

d) $$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) = \tan^{-1}\left(\frac{1.45}{1}\right) = 55.41^\circ$$

$$\theta = 90^\circ - \theta_B = 34.59^\circ$$

e) Fractional power transmitted $= \left(1 - e^{-2(r/w)^2}\right)\Big|_{r = D/2}$

Fractional loss $= e^{-2(r/w)^2}\Big|_{r=D/2}$

Best case, $W = W_0$. Fractional loss $= e^{-2\left(\frac{2}{0.2953}\right)^2} = 1.437 \times 10^{-40}$

Worst case, $w = w_2$. Fractional loss $= e^{-2\left(\frac{2}{0.590d}\right)^2} = 1.095\times10^{-10}$ very little loss.

f) $\nu = \frac{c}{2nd}p + \frac{(1+m+n)}{\pi}\frac{c}{2nd}\tan^{-1}\left(\frac{d}{z_0}\right)$ since $z_1 = 0$, $z_2 = d$

$\nu = 2\times10^8\left(p + \frac{(1+m+n)}{\pi}\frac{\pi}{3}\right) = 2\times10^8\left(p + \frac{1+m+n}{3}\right)$

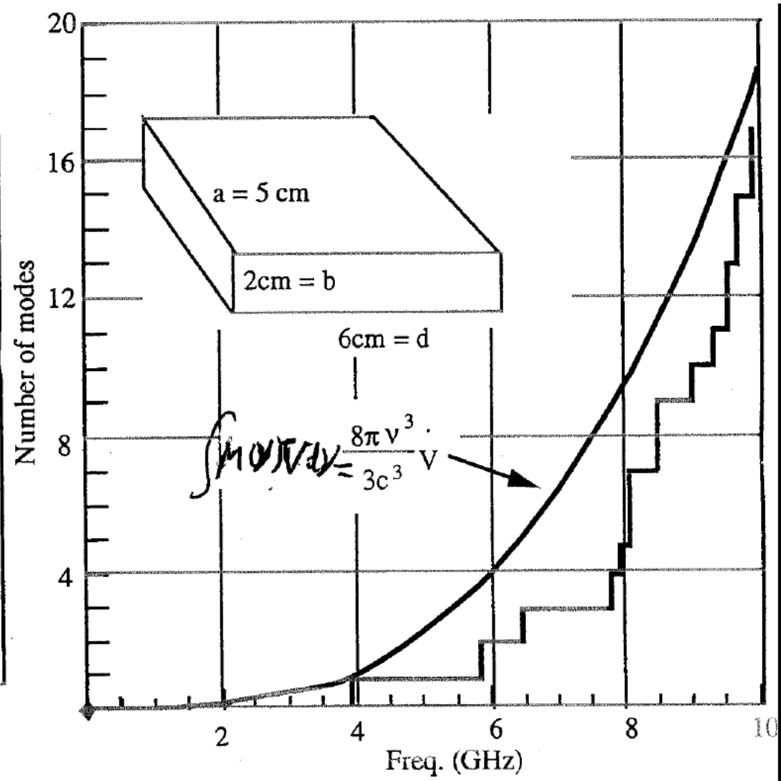
5) Exercise 10.3-1

a) $\#\text{ of mode} = \underset{\text{polarization}}{2} \times \frac{\frac{1}{4}\pi\left(\frac{2\pi\nu}{c}\right)^2}{(\pi/d)^2} = 2\pi\left(\frac{\nu}{c}\right)^2 d^2$

$M(\nu) = \frac{1}{\text{Area}}\frac{d\,\#\text{ of mode}}{d\nu} = \frac{1}{\cancel{d^2}}4\pi\frac{\nu}{c^2}\cancel{d^2} = \frac{4\pi\nu}{c^2}$

6) $\nu = \frac{c}{2}\left\{\left(\frac{q_x}{a}\right)^2 + \left(\frac{q_y}{b}\right)^2 + \left(\frac{q_z}{d}\right)^2\right\}^{1/2}$

q_x	q_y	q_z	ν (GHz)	mode	Σ
1	0	1	3.91	TE_{101}	1
1	0	2	5.83	TE_{102}	2
2	0	1	6.50	TE_{201}	3
2	0	2	7.81	TE_{202}	4
0	1	1	7.96	TE_{011}	5
1	0	3	8.08	TE_{013}	6
1	1	0	8.08	TM_{110}	7
1	1	1	8.46	TE,TM_{111}	9
0	1	2	9.01	TE_{012}	10
3	0	1	9.34	TE_{301}	11
1	1	2	9.5	TE,TM_{112}	13
2	0	3	9.61	TE	14
2	1	0	9.61	TM	15
2	1	1	9.93	TE,TM	17



7) a) $\phi(z=l) = kl - (1+m+n)\tan^{-1}\left(\frac{l}{z_0}\right)$, $\phi(z=d+l) = k(d+l) - (1+m+n)\tan^{-1}\left(\frac{l+d}{z_0}\right)$

Resonant condition: $\phi(z=l+d) - \phi(z=l) = \pi p$

$\Rightarrow kd - (1+m+n)\left[\tan^{-1}\left(\frac{l+d}{z_0}\right) - \tan^{-1}\left(\frac{l}{z_0}\right)\right] = \pi p$ (note: $k = \frac{\omega}{c} = \frac{2\pi\nu}{c}$)

$\nu = \frac{c}{2d}\left[p + \frac{(1+m+n)}{\pi}\left(\tan^{-1}\left(\frac{l+d}{z_0}\right) - \tan^{-1}\left(\frac{l}{z_0}\right)\right)\right]$

b) $\nu_{1,2,n} - \nu_{0,0,n} = \frac{c}{2d}\left[1 + \frac{2}{\pi}\left(\tan^{-1}\left(\frac{100}{125}\right) - \tan^{-1}\left(\frac{25}{125}\right)\right)\right]$

(Assume $n=1$) $= 2\times10^8\left(1 + \frac{2}{\pi}(0.6747 - 0.1974)\right)$

$$= 2\times10^8\left(1+2\times\frac{27.35}{180}\right) = 3.216\times10^8\ (Hz)$$

c) $|R_1| = z\left(1+\left(\frac{z_0}{z}\right)^2\right)\Big|_{z=0.25} = 6.5(m)$ $R_1 = -6.5(m)$ (diverging mirror)

$|R_2| = z\left(1+\left(\frac{z_0}{z}\right)^2\right)\Big|_{z=0.1} = 2.5625(m)$ $R_2 = 2.5625(m)$ (converging mirror)

For b), you can interpret n is the index for longitudinal..

$$\nu_{1,2,n}-\nu_{0,0,n} = \frac{c}{2d}\left[(n-n)+\frac{(1+2+1-0-0-1)}{\pi}(0.6747-0.1974)\right]$$

$$= 2\times10^8\left(3\times\frac{27.35}{180}\right) = 9.117\times10^7\ (Hz)$$