HW9 Solution

1) $E_n = -\frac{Z^2 E_0}{n^2}$ (Eq. 3.13) in Laser Fundamentals

where $E_0 = \frac{m e e^4}{8\epsilon_0^2 h^2} = 13.595\,eV$ (Eq. 3.9) and C has $Z=6$.

$E_n = -\frac{489.4}{n^2}$ (eV) ionization potential $= E_1 = -489.4$ (eV) (only one electron remains)

$h\nu_{32} = E_3 - E_2 = 489.4\left(\frac{1}{4} - \frac{1}{9}\right) = 67.97$ (eV)

$\lambda_{32}^{(\mu m)} = \frac{1.24}{E_3 - E_2} = 0.01824\,(\mu m) = 18.24\,(nm)$

2) a) $I_{out} = (e^{\gamma L})^6 I_{in} = (e^{0.015\times 50})^6 = (2.117)^6 = 90.02\,pW/cm^2$.

b) $I_{out} = R^3\,(e^{\gamma L})^6 I_{in} = \left(\sqrt{R}\;e^{\gamma L}\right)^6 I_{in} = \left(\sqrt{0.8}\,e^{0.015\times 50}\right)^6 = 1.895^6 = 46.09 pW/cm^2$.

c) $I_{tran} = (1-R)(e^{\gamma L})[1 + \left(\sqrt{R}\;e^{\gamma L}\right)^2 + \left(\sqrt{R}\;e^{\gamma L}\right)^4]\,I_{in} = 7.384 pW/cm^2$

3 a) $\nu_0 = ?$ $\Delta\nu = ?$ $\lambda_0 = 5000\,Å$, $\Delta\lambda = 1\,Å$, $n=1$, $V = 2\,cm^3$

$\nu_0 = \frac{c}{\lambda_0} = \frac{3\times10^8}{5000\times10^{-10}} = 6\times10^{14}$ (Hz) or $\frac{1}{\lambda} = \frac{1}{5\times10^{-5}} = 2\times10^4$ (cm^{-1})

$\Delta\nu = \nu_0\frac{\Delta\lambda}{\lambda_0} = 6\times10^{14}\,\frac{1}{5000} = 1.2\times10^{11}$ (Hz) or $\frac{1.2\times10^{11}}{3\times10^8}\,\frac{1}{100} = 4$ (cm^{-1})

b) Number of modes $\simeq M(\nu)\,V\,d\nu = \frac{8\pi\nu_0^2}{c^3}\Delta\nu\,V$

$= \frac{8\pi(6\times10^{14})^2}{(3\times10^8)^3}\,1.2\times10^{11}\times 2\times10^{-6}$

$= 8.042\times10^{10}$

c) $d = 20\,cm$ $\Delta\nu_{FSR} = \frac{c}{2nd} = \frac{3\times10^8}{2(0.2)} = 7.5\times10^8$

Number of TEM_{00} modes $= 2\times\frac{1.2\times10^{11}}{7.5\times10^8} = 320$

d) Probability of spontaneous photon appearing in one of the polarized TEM_{00} modes $= \frac{320}{8.04\times10^{10}} = 3.979\times10^{-9}$

e) $t_{sp}^{-1} = 10^7\,(s^{-1})$

$\sigma(\nu) = \frac{\lambda_0^2}{8\pi n^2 t_{sp}}\,g(\nu) = \frac{0.25(10^{-6})^2}{8\pi}\,10^7 g(\nu) = 9.948\times10^{-8}\,g(\nu)\,(m^2)$

Assume Lorentzian

$\sigma(\nu_0) = 9.948\times10^{-8}\,\frac{2}{\Delta\nu\pi} = 5.278\times10^{-19}\,(m^2)$, $\sigma(\nu) = \frac{1900}{(\nu - 6\times10^{14})^2 + (6\times10^{10})^2}$

4) Problem 13.3-3

$\lambda_o = 0.7\mu m$, $t_{sp} = 3ms$, $\Delta\nu = 50GHz$, Lorentzian lineshape

$V = 100cm^3$, $n = 1$. 2 modes (ν_o, $\nu_o + \Delta\nu$), each mode is excited with 1000 photons.

$$P_{st} = n\frac{c\sigma(\nu)}{V} = \frac{n\,c}{V}\frac{\lambda_o^2}{8\pi n^2 t_{sp}} g(\nu) = \frac{1000 \times 3\times10^8}{100\times10^{-6}} \frac{(0.7\times10^{-6})^2}{8\pi\ 3\times10^{-3}} g(\nu)$$

$$= 19496 \frac{\Delta\nu/2\pi}{(\nu-\nu_o)^2 + (\Delta\nu/2)^2} = \frac{1.551\times10^{14}}{(\nu - 4.286\times10^{14})^2 + (25\times10^9)^2}$$

spon. emission rate $= 1/t_{sp} = 1/3\times10^{-3}\ (s^{-1})$

stimulated emission rate $= P_{st}(\nu_o) + P_{st}(\nu_o+\Delta\nu) = \frac{1.551\times10^{14}}{(25\times10^9)^2}(1+\frac{1}{5})$

$= 2.978\times10^{-7}\ (s^{-1})$

Lifetime for $N_2 = \frac{1}{\Sigma \text{ rates}} \simeq 3\ (ms)$ since very small simulated emission probability.

stimulated emiss rate = spon. emiss. rate $\Rightarrow \frac{1}{3\times10^{-3}} = \frac{2.978\times10^{-7} n}{1000} \Rightarrow n = 1.119\times10^{12}$

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5) Normalized paraxial wave equation:

$$\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N \frac{\partial A}{\partial r_N}\right) + \frac{1}{r_N^2}\frac{\partial^2 A}{\partial\phi^2} - 2j\frac{\partial A}{\partial Z} = 0, \quad r_N = \frac{r}{w_0}, \quad Z = \frac{z}{2z_0}$$

Let $A = B(r_N, Z)\, e^{\pm jl\phi}$ and $B(r_N, Z) = g\left(\left(\frac{\sqrt{2}\, r_N}{\tilde{w}(Z)}\right)^2\right)\tilde{Z}(Z)$

$* A_G(r_N, Z)$, $\tilde{w}(Z) = [1 + (2Z)^2]^{\frac{1}{2}}$

$A_G(r_N, Z)$ is the Gaussian beam solution, i.e.

$$\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A_G}{\partial r_N}\right) - 2j\frac{\partial A_G}{\partial Z} = 0 \quad — (1)$$

$$\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A}{\partial r_N}\right) + \frac{1}{r_N^2}\frac{\partial^2 A}{\partial\phi^2} - 2j\frac{\partial A}{\partial Z} = 0$$

$$\Rightarrow \frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A}{\partial r_N}\right) - \frac{l^2}{r_N^2}A - 2j\frac{\partial A}{\partial Z} = 0$$

$$\frac{\partial A}{\partial r_N} = \left(g'\, 2\left(\frac{\sqrt{2}\, r_N}{\tilde{w}}\right)\frac{\sqrt{2}}{\tilde{w}} A_G + g\frac{\partial A_G}{\partial r_N}\right)\tilde{Z}\, e^{\pm jl\phi}$$

$$r_N\frac{\partial A}{\partial r_N} = \left(g'\left(\frac{2}{\tilde{w}}\right)^2 r_N^2 A_G + g\, r_N\frac{\partial A_G}{\partial r_N}\right)\tilde{Z}\, e^{\pm jl\phi}$$

$$\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A}{\partial r_N}\right) = \Big[g''\left(\frac{2}{\tilde w}\right)^2\left(\frac{2}{\tilde w}\right)^2 r_N r_N^2 A_G + g'\left(\frac{2}{\tilde w}\right)^2\left(2r_N A_G + r_N^2\frac{\partial A_G}{\partial r_N}\right)$$
$$+ g'\left(\frac{2}{\tilde w}\right)^2 r_N^2\frac{\partial A_G}{\partial r} + g\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A_G}{\partial r_N}\right)\Big]\tilde Z\, e^{\pm jl\phi}$$

$$\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A}{\partial r_N}\right) = \Big[g''\left(\frac{2}{\tilde w}\right)^4 r_N^2 A_G + g'\left(\frac{2}{\tilde w}\right)^2\left(2A_G + r_N\frac{\partial A_G}{\partial r_N}\right)$$
$$+ g'\left(\frac{2}{\tilde w}\right)^2 r_N\frac{\partial A_G}{\partial r} + g\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A_G}{\partial r_N}\right)\Big]\tilde Z\, e^{\pm j\phi}$$
$$= \Big[g''\left(\frac{2}{\tilde w}\right)^4 r_N^2 A_G + 2g'\left(\frac{2}{\tilde w}\right)^2\left(A_G + r_N\frac{\partial A_G}{\partial r_N}\right) + g\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A_G}{\partial r_N}\right)\Big]$$
$$\tilde Z\, e^{\pm jl\phi}$$

Let $u = \left(\frac{\sqrt{2}\,r_N}{\tilde w}\right)^2 = \frac{2r_N^2}{1+(2z)^2}$

$$\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A}{\partial r_N}\right) - \frac{l^2}{r_N^2}A - 2j\frac{\partial A}{\partial z} = 0$$

$$\Rightarrow \Big[g''\frac{2^3}{\tilde w^2}uA_G + 2g'\left(\frac{2}{\tilde w}\right)^2\left(A_G + r_N\frac{\partial A_G}{\partial r_N}\right) + g\frac{1}{r_N}\frac{\partial}{\partial r_N}\left(r_N\frac{\partial A_G}{\partial r_N}\right)\Big]\tilde Z$$
$$e^{\pm jl\phi} - \frac{l^2}{r_N^2}gA_G\tilde Z\, e^{\pm jl\phi} - 2j\Big(g'\,2r_N^2\left(\frac{-1}{1+(2z)^2}\right)4z\,\tilde Z A_G$$
$$+ g\tilde Z' A_G + g\tilde Z\frac{\partial A_G}{\partial z}\Big)e^{\pm jl\phi} = 0$$

Apply ① and eliminate common term $e^{\pm jl\phi}$,

$$\Big[g''\frac{8}{\tilde w^2}uA_G + 2g'\left(\frac{2}{\tilde w}\right)^2\left(A_G + r_N\frac{\partial A_G}{\partial r_N}\right)\Big]\tilde Z - \frac{l^2}{r_N^2}gA_G\tilde Z - 2j\Big(-g'\frac{u}{\tilde w^2}8z$$
$$\tilde Z A_G + g\tilde Z' A_G\Big) = 0 \qquad - ②$$

$$②/B: \quad \frac{g''}{g}\frac{8}{\tilde w^2}u + 2\frac{g'}{g}\left(\frac{2}{\tilde w}\right)^2\left(1 + \frac{r_N}{A_G}\overset{-j\frac{r_N}{z+\frac{1}{2}j}A_G}{\frac{\partial A_G}{\partial r_N}}\right) - \overset{\frac{2}{u\tilde w^2}}{\cancel{\frac{1}{r_N^2}}}l^2 - 2j\Big(-\frac{g'}{g}\frac{u}{\tilde w^2}8z$$
$$+ \frac{\tilde Z'}{\tilde Z}\Big) = 0 \qquad - ③$$

Consider $\frac{g'}{g}$ terms in ③:

$$2\frac{g'}{g}\left(\frac{2}{\tilde w}\right)^2\left(1 + \frac{-jr_N^2}{z+\frac{1}{2}j}\right) + 2j\frac{g'}{g}\frac{u}{\tilde w^2}8z = 2\frac{g'}{g}\left(\frac{2}{\tilde w}\right)^2\left[1 - \frac{2jr_N^2}{2z+j} + 2juz\right]$$

$$= 2\frac{g'}{g}\left(\frac{2}{\tilde{w}}\right)^2\left[1 - j\frac{u\tilde{w}^2}{\tilde{w}^2}(2z\cdot j) + 2juz\right] = \frac{2g'}{g}\left(\frac{2}{\tilde{w}}\right)^2(1-u)$$

③ becomes:

$$\frac{g''}{g}\frac{8}{\tilde{w}^2}u + \frac{2g'}{g}\left(\frac{2}{\tilde{w}}\right)^2(1-u) - \frac{2}{u\tilde{w}^2}l^2 - 2j\frac{\tilde{Z}'}{\tilde{Z}} = 0$$

$$\frac{g''}{g}8u + 8\frac{g'}{g}(1-u) - \frac{2}{u}l^2 + 8m + 4l - 2j\frac{\tilde{Z}'}{\tilde{Z}} - 8m - 4l = 0 \quad . \quad -④$$

we separate ④ into 2 independent differential eqns:

$$4ug'' + 4g'(1-u) + \left(-\frac{l^2}{u} + 4m + 2l\right)g = 0 \qquad -⑤$$

$$-j\frac{\tilde{Z}'}{\tilde{Z}} = 4m + 2l \qquad -⑥$$

Find solution for ⑤: Let $g = u^{l/2}L$

$$g' = \frac{l}{2}u^{l/2-1}L + u^{l/2}L', \quad g'' = \frac{l}{2}\left(\frac{l}{2}-1\right)u^{l/2-2}L + \frac{l}{2}u^{\frac{l}{2}-1}L' + \frac{l}{2}u^{l/2-1}L' + u^{l/2}L''$$

$$= \frac{l}{2}\left(\frac{l}{2}-1\right)u^{l/2-2}L + lu^{\frac{l}{2}-1}L' + u^{\frac{l}{2}}L''$$

⑤ becomes: $4u\left(\frac{l}{2}\left(\frac{l}{2}-1\right)u^{l/2-2}L + u^{l/2-1}L' + u^{l/2}L''\right) + 4(1-u)\left(\frac{l}{2}u^{l/2-1}L + u^{l/2}L'\right) + \left(4m + 2l - \frac{l^2}{u}\right)u^{l/2}L = 0$

$$\Rightarrow 4u^{l/2+1}L'' + \left(4lu^{l/2} + 4(1-u)u^{l/2}\right)L' + \left[4u\frac{l}{2}\left(\frac{l}{2}-1\right)u^{-2} + 4(1-u)\frac{l}{2}u^{-1} + 4m + 2l - \frac{l^2}{u}\right]u^{l/2}L = 0$$

$$\Rightarrow 4uL'' + 4(l+1-u)L' + \left\{\left[2(\cancel{1}-\cancel{u})l + 2l\left(\cancel{\tfrac{l}{2}}-\cancel{1}\right)\right]u^{-1} + 4m + \cancel{2l} - \cancel{\tfrac{l^2}{u}}\right\}L = 0$$

$\Rightarrow uL'' + (l+1-u)L' + mL = 0$ This is the generalized Laguerre differential eqn which has solution in terms of generalized Laguerre polynomials $L_m^l(u)$

Find solution for ⑥: Let $\tilde{Z} = e^{jv(z)}$

$$\tilde{Z}' = je^{jv(z)}v'(z) = j\tilde{Z}v'(z)$$

⑥ becomes: $\tilde{w}^2 v'(z) = 4m + 2l$

$$\Rightarrow (1 + (2z)^2)\frac{dv}{dz} = 4m + 2l$$

$$\Rightarrow dv = (4m + 2l)\frac{dz}{1 + (2z)^2} \qquad (\text{let } X = 2z)$$

$$\Rightarrow \quad v = \frac{(4m+2l)}{2}\int \frac{dx}{1+x^2} \Rightarrow v = (2m+l)\tan^{-1}(x)$$

$$\Rightarrow v = (2m+l)\tan^{-1}(2z)$$

Hence $A = \left(\frac{\sqrt{2}\,r_N}{\tilde{w}(z)}\right)^l L_m^l\left(\left(\frac{\sqrt{2}\,r_N}{\tilde{w}(z)}\right)^2\right) e^{j((2m+l)\tan^{-1}(2z))} e^{\pm jl\phi}$

$A_G(r_N, z)$

$$= A_{l,m}\left(\frac{r_N}{\tilde{w}(z)}\right)^l L_m^l\left(\left(\frac{\sqrt{2}\,r_N}{\tilde{w}(z)}\right)^2\right) e^{\pm jl\phi}$$

$$e^{j\left((2m+l+1)\tan^{-1}(2z) - \frac{r_N^2\, 2z}{\tilde{w}(z)^2}\right)} \quad \frac{e^{-\left(\frac{r_N}{\tilde{w}(z)}\right)^2}}{\tilde{w}(z)}$$