14.8 HIGHER-ORDER GAUSSIAN BEAM MODES

The Gaussian beam of Table 14.1 results from the solution (14.5.22) of the paraxial wave equation (14.4.19). However, this is not the only solution. In this section we will consider a more general type of Gaussian-beam solution of the paraxial wave equation. The Gaussian beam of Table 14.1 will emerge as a special case of this more general solution.

We arrived at the solution (14.5.22) of the paraxial wave equation by guessing a solution of the form (14.5.2). In attempting to obtain other solutions we will proceed in a similar fashion, assuming a solution of the form

\[
\mathcal{E}_0(r) = Ag \left( \frac{x}{w(z)} \right) h \left( \frac{y}{w(z)} \right) e^{i\phi(z)} e^{i(kx^2 + y^2)/2q(z)}
\]

(14.8.1)

We assume \(w(z)\) and \(q(z)\) are the same as before, i.e., that the spot size and radius of curvature of our more general Gaussian beam are given in Table 14.1. In fact if

\[P(z) = p(z)\]

and

\[g \left( \frac{x}{w(z)} \right) = h \left( \frac{y}{w(z)} \right) = 1\]

then the trial solution (14.8.1) reduces exactly to (14.5.2). The fact that \(g\) and \(h\) are functions of \(x/w(z)\) and \(y/w(z)\), respectively, means that the intensity pattern associated with (14.8.1) will scale according to the spot size \(w(z)\). This intensity pattern will be a function of \(x/w(z)\) and \(y/w(z)\), as is the intensity pattern given in Table 14.1. Our task is to find \(g\), \(h\), and \(P\) such that (14.8.1) satisfies the paraxial wave equation.

Using our trial solution (14.8.1) in the paraxial wave equation (14.4.19), we obtain differential equations for \(g\), \(h\), and \(P\). Since the algebra is straightforward but rather tedious, we will omit the details of the derivation and give only the main steps. First we use the fact that \(g\) and \(h\) are functions of the independent variables \(\xi\) and \(\eta\), and for \(z\) unless each of these terms is separately constant. Therefore we write

\[
\frac{1}{g(\xi)} \left( \frac{d^2g}{d\xi^2} - 4\xi \frac{dg}{d\xi} \right) + \frac{1}{h(\eta)} \left( \frac{d^2h}{d\eta^2} - 4\eta \frac{dh}{d\eta} \right) + \left( \frac{2ik}{q(z)} - 2k \frac{dP}{dz} \right) w^2(z) = 0
\]

(14.8.4)

after division by \(g(\xi) h(\eta)\). The functions \(g(\xi), h(\eta),\) and \(P(z)\) must satisfy this equation in order for (14.8.1) to satisfy the paraxial wave equation.

Now the first term on the left-hand side of (14.8.4) is a function only of the independent variable \(\xi\), the second term is a function only of the independent variable \(\eta\), and the third term is a function only of the independent variable \(z\). Thus Eq. (14.8.4) cannot hold for all values of the independent variables \(\xi\), \(\eta\), and \(z\) unless each of these terms is separately constant. Therefore we write

\[
\frac{1}{g(\xi)} \left( \frac{d^2g}{d\xi^2} - 4\xi \frac{dg}{d\xi} \right) = -a_1
\]

(14.8.5)

\[
\frac{1}{h(\eta)} \left( \frac{d^2h}{d\eta^2} - 4\eta \frac{dh}{d\eta} \right) = -a_2
\]

(14.8.6)

and

\[
\left( \frac{2ik}{q(z)} - 2k \frac{dP}{dz} \right) w^2(z) = a_1 + a_2
\]

(14.8.7)

where \(a_1\) and \(a_2\) are constants. Thus we have reduced the problem of solving the partial differential equation (14.4.19) in three independent variables to the problem of solving the three ordinary differential equations (14.8.5)–(14.8.7). This is another example of the method of “separation of variables,” which was also used in Chapter 2.
It is convenient to write (14.8.5) in a slightly different form by defining the new variable

\[ u = \sqrt{2} \xi \]  \hspace{1cm} (14.8.8)

Since

\[ \frac{dg}{d\xi} = \frac{dg}{du} \frac{du}{d\xi} = \sqrt{2} \frac{dg}{du} \]  \hspace{1cm} (14.8.9a)

and

\[ \frac{d^2g}{d\xi^2} = 2 \frac{d^2g}{du^2} \]  \hspace{1cm} (14.8.9b)

we have

\[ \frac{d^2g}{du^2} - 2u \frac{dg}{du} + \frac{a_1}{2} g = 0 \]  \hspace{1cm} (14.8.10)

The reason we have chosen to write (14.8.5) in this form is that the equation (14.8.10) arises in many different problems. It appears, for example, in the quantum mechanics of the harmonic oscillator, as described in Problem 5.8. A solution of (14.8.10) stays finite as \( u \to \infty \) only if the constant \( a_1 \), satisfies

\[ \frac{a_1}{2} = 2m, \hspace{1cm} m = 0, 1, 2, \ldots \]  \hspace{1cm} (14.8.11)

The allowed (finite) solutions of (14.8.10) are the Hermite polynomials, the first few of which are listed in Appendix 5.B. The allowed solutions for the function \( g \) in our trial solution (14.8.1) are thus

\[ g \left( \frac{x}{w(z)} \right) = H_m \left( \sqrt{2} \frac{x}{w(z)} \right), \hspace{1cm} m = 0, 1, 2, \ldots \]  \hspace{1cm} (14.8.12)

In a similar fashion we obtain the allowed solutions

\[ h \left( \frac{y}{w(z)} \right) = H_n \left( \sqrt{2} \frac{y}{w(z)} \right), \hspace{1cm} n = 0, 1, 2, \ldots \]  \hspace{1cm} (14.8.13)

for the function \( h \).

It remains to determine \( P(z) \). Using Eqs. (14.5.11), (14.5.17), and (14.5.18), we obtain from (14.8.7) the differential equation

\[ \frac{dP}{dz} = \frac{iz}{z^2 + z_0^2} \left( \frac{m + n + 1}{z^2 + z_0^2} \right) \]  \hspace{1cm} (14.8.14)

which may be integrated to give

\[ P(z) = i \ln \left( \frac{z^2 + z_0^2}{z^2 + z_0^2} \right) = (m + n + 1) \phi(z) \]  \hspace{1cm} (14.8.15)

or

\[ e^{iP(z)} = \frac{e^{-i(m+n+1)\phi(z)}}{\sqrt{1 + z^2/z_0^2}} = \frac{w_0}{w(z)} e^{-i(m+n+1)\phi(z)} \]  \hspace{1cm} (14.8.16)

where \( \phi(z) = \tan^{-1} \left( z/z_0 \right) \), as in (14.5.21). Collecting the results (14.8.12), (14.8.13), and (14.8.16), we have a solution (14.8.1) to the paraxial wave equation. The electric field is thus

\[ E_{nm}(x, y, z) = \frac{Aw_0}{w(z)} H_m \left( \sqrt{2} \frac{x}{w(z)} \right) H_n \left( \sqrt{2} \frac{y}{w(z)} \right) \times e^{i[kz-(m+n+1)\tan^{-1}z/z_0]} \times e^{i(k^2z^2+y^2)/2R(z)} e^{-(z^2+y^2)/w(z)} \]  \hspace{1cm} (14.8.17)

Note that when \( m = n = 0 \) we recover the solution (14.5.22), so our previous Gaussian-beam solution is therefore the "lowest-order" or "zero-order" case of (14.8.17). Another important point is that \( R(z) \) and \( w(z) \) are independent of \( m \) and \( n \); all higher-order Gaussian beams are characterized by the same values of \( R(z) \) and \( w(z) \) as the lowest-order one. Furthermore, all higher-order Gaussian beams satisfy the same \( ABCD \) law as the lowest-order one: a Gaussian beam of order \((m, n)\) remains a Gaussian beam of the same order after propagation in free space or transformation by a thin lens or a spherical mirror, but its \( q \) parameter is changed according to the \( ABCD \) law.

Because they have the same spot size and radius of curvature as the lowest-order beam, the higher-order Gaussian beams also form modes of stable resonators satisfying (14.7.20). All the properties of Table 14.2 apply as well to such higher-order modes, except for the resonance frequencies. Following exactly the same approach that led to (14.7.14), we obtain for a Gaussian mode of order \((m, n)\) the allowed mode frequencies

\[ \nu_{qmn} = \frac{c}{2L} \left( q + \frac{1}{\pi} (m + n + 1) \cos^{-1} \sqrt{g_1 g_2} \right) \]  \hspace{1cm} (14.8.18)

with \( q \) a positive integer or zero, and the sign convention of (14.7.14).